



TITLE:

範囲基準下の最適停止について (不 確実性と意思決定数理の諸問題)

AUTHOR(S):

Tsurusaki, Kazuyoshi; Iwamoto, Seiichi

CITATION:

Tsurusaki, Kazuyoshi ...[et al]. 範囲基準下の最適停止について (不確実性と意思決定数理の諸問題). 数理解析研究所講究録 2004, 1373: 229-237

ISSUE DATE:

2004-05

URL:

<http://hdl.handle.net/2433/25547>

RIGHT:

範囲基準下の最適停止について

長崎大学経済学部 津留崎 和義 (Kazuyoshi Tsurusaki)
Faculty of Economics, Nagasaki Univ.

九州大学大学院経済学研究院 岩本 誠一 (Seiichi Iwamoto)
Graduate School of Economics, Kyushu Univ.

1 Introduction

In this paper, we consider the optimal stopping problem for compound criteria, whose counterpart is simple criteria such as terminal, additive and minimum. We introduce a new notion of gain process, which is evaluated at stopped state. Some of gain processes are terminal, additive, minimum, range, ratio, sample variance. The former three are simple. The latter three are compound. In this paper we discuss the compound criterion such as range, mid-range, ratio, average and sample variance.

2 General Process

We consider a class of finite-stage optimal stopping problems from a view point of reward accumulation. An N -stage problem has to stop by time N at the latest. Each stage allows either stop or continue. When a decision maker stops on a state at n -th stage, she gets a reward which is closely related to all the states she has experienced.

Let $\{X_n\}_0^N$ be a Markov chain on a finite state space X with a transition law $p = \{p(\cdot|\cdot)\}$. Letting $X^k := X \times X \times \cdots \times X$ (k times) be the direct product of k state spaces X , we take $H_n := X^{n+1}$; the set of all *subpaths* $h_n = x_0 x_1 \cdots x_n$ up to stage n :

$$H_n = \{h_n = x_0 x_1 \cdots x_n \mid x_m \in X, 0 \leq m \leq n\} \quad 0 \leq n \leq N.$$

In particular, we set

$$\Omega := H_N.$$

Its element $\omega = h_N = x_0 x_1 \cdots x_N$ is called a *path*.

Let \mathcal{F}_m^n be the set of all subsets in Ω which are determined by random variables $\{X_m, X_{m+1}, \dots, X_n\}$, where $X_k : \Omega \rightarrow X$ is the projection, $X_k(\omega) = x_k$. Strictly, \mathcal{F}_m^n is the σ -field on Ω generated by the set of all subsets of the form

$$\{X_m = x_m, X_{m+1} = x_{m+1}, \dots, X_n = x_n\} \quad (\subset \Omega)$$

where x_m, x_{m+1}, \dots, x_n are all elements in state space X . Let us take $\mathbb{N} = \{0, 1, \dots, N\}$. A mapping $\tau : \Omega \rightarrow \mathbb{N}$ is called a *stopping time* if

$$\{\tau = n\} \in \mathcal{F}_0^n \quad \forall n \in \mathbb{N}.$$

where $\{\tau = n\} = \{x_0 x_1 \cdots x_N \mid \tau(x_0 x_1 \cdots x_N) = n\}$. The stopping time τ is called $\{\mathcal{F}_0^n\}_0^N$ -*adapted*. Let \mathcal{T}_0^N be the set of all such stopping times. Any stopping time $\tau \in \mathcal{T}_0^N$ generates a *stopped subhistory* (random variable) $(X_0, X_1, \dots, X_{\tau-1}, X_\tau)$ on Ω through

$$X_{\tau-n}(\omega) := X_{\tau(\omega)-n}(\omega) \quad 0 \leq n \leq \tau(\omega).$$

Let $\{g_n\}_0^N$ be a sequence of *gain functions*

$$g_n : H_n \rightarrow R^1 \quad 0 \leq n \leq N.$$

Then a *gain process* $\{G_n\}_0^N$ is defined by

$$G_n := g_n(X_0, X_1, \dots, X_n).$$

Then any stopping time τ yields a *stopped reward* (random variable) $G_\tau : \Omega \rightarrow R^1$:

$$G_\tau(\omega) = G_{\tau(\omega)}(X_0(\omega), X_1(\omega), \dots, X_{\tau-1}(\omega), X_\tau(\omega)).$$

We remark that the expected value $E_{x_0}[G_\tau]$ is expressed by sum of multiple sums :

$$\begin{aligned} E_{x_0}[G_\tau] &= \sum_{n=0}^N \sum_{\{\tau=n\}} G_n(h_n) P_{x_0}(X_0 = x_0, \dots, X_n = x_n) \\ &= \sum_{n=0}^N \sum_{\{\tau=n\}} G_n(h_n) p(x_1|x_0) p(x_2|x_1) \cdots p(x_n|x_{n-1}). \end{aligned}$$

Now we consider the optimal stopping problem for the gain process :

$$G_0(x_0) \quad \text{Max} \quad E_{x_0}[G_\tau] \quad \text{s.t.} \quad \tau \in \mathcal{T}_0^N. \quad (1)$$

Then we have the corresponding recursive equation and optimal stopping time ([4]) :

Theorem 2.1

$$\begin{cases} v_N(h) = g_N(h) & h \in H_N \\ v_n(h) = \text{Max} [g_n(h), E_x[v_{n+1}(h, X_{n+1})]] \\ h = (x_0, \dots, x_{n-1}, x) \in H_n, \quad 0 \leq n \leq N-1. \end{cases} \quad (2)$$

Theorem 2.2 The stopping time τ^* :

$$\tau^*(\omega) = \min\{n \geq 0 : v_n(h_n) = G_n(h_n)\} \quad \omega = x_0 x_1 \cdots x_N$$

is optimal :

$$E_{x_0}[G_{\tau^*}] \geq E_{x_0}[G_\tau] \quad \forall \tau \in \mathcal{T}_0^N.$$

3 Expanded Control Chain

Now, in this section, let us discuss a general result for range process. We consider a maximization problem of expected value for stopped process under *range* criterion (As for nonstopping but control problems, see [11–16, 21]).

Let $\{X_n\}_0^N$ be the Markov chain on the finite state space X with the transition law $p = \{p(\cdot|\cdot)\}$ (Section 2). Let $g_n : X \rightarrow R^1$ be a *stop reward* for $0 \leq n \leq N$ and $r_n : X \rightarrow R^1$ be a *continue reward* for $0 \leq n \leq N-1$. Then an accumulation process is constructed as follows. When a decision-maker stops at stage x_n on stage n through a subhistory $(x_0, x_1, \dots, x_{n-1})$, he or she will incur the range of reward up to stage n :

$$R_{an}(h_n) := r_0 \vee r_1 \vee \cdots \vee r_{n-1} \vee g_n - r_0 \wedge r_1 \wedge \cdots \wedge r_{n-1} \wedge g_n$$

where

$$h_n = (x_0, x_1, \dots, x_n), \quad r_m = r_m(x_m), \quad g_n = g_n(x_n).$$

The accumulation process $\{R_{a_n}\}_0^N$ is called a *range process*. Thus a stopped reward by adopting stopping time τ for range process is

$$R_{a\tau} = r_0 \vee r_1 \vee \dots \vee r_{\tau-1} \vee g_\tau - r_0 \wedge r_1 \wedge \dots \wedge r_{\tau-1} \wedge g_\tau.$$

Now we consider the optimal stopping problem for range process :

$$\text{Ra}_0(x_0) \quad \text{Max} \quad E_{x_0}[R_{a\tau}] \quad \text{s.t.} \quad \tau \in \mathcal{T}_0^N.$$

The expected value of range is the sum of multiple sums :

$$E_{x_0}[R_{a\tau}] = \sum_{n=0}^N \sum_{\{\tau=n\}} \{R_{a_n}(h_n) \times p(x_1|x_0)p(x_2|x_1) \cdots p(x_n|x_{n-1})\}.$$

Let us now imbed $\text{Ra}_0(x_0)$ into a new class of additional parametric subproblems [2, 17]. First we define the *past-valued (cumulative) random variables* $\{\tilde{\Lambda}_n\}_0^N, \{\tilde{\Xi}_n\}_0^N$ up to n -th stage and the *past-value sets* $\{\Lambda_n\}_0^N, \{\Xi_n\}_0^N$ they take :

$$\begin{aligned} \tilde{\Lambda}_0 &:= \tilde{\lambda}_0 \quad \text{where } \tilde{\lambda}_0 \text{ is smaller than or equal to } g_n(x), r_n(x) \\ \tilde{\Xi}_0 &:= \tilde{\xi}_0 \quad \text{where } \tilde{\xi}_0 \text{ is larger than or equal to } g_n(x), r_n(x) \\ \tilde{\Lambda}_n &:= r_0(X_0) \vee \dots \vee r_{n-1}(X_{n-1}) \\ \tilde{\Xi}_n &:= r_0(X_0) \wedge \dots \wedge r_{n-1}(X_{n-1}) \\ \Gamma_0 &:= \{(\tilde{\lambda}_0, \tilde{\xi}_0)\} \\ \Gamma_n &:= \left\{ (\lambda_n, \xi_n) \mid \begin{array}{l} \lambda_n = r_0(x_0) \vee \dots \vee r_{n-1}(x_{n-1}), \\ \xi_n = r_0(x_0) \wedge \dots \wedge r_{n-1}(x_{n-1}), \\ (x_0, \dots, x_{n-1}) \in X \times \dots \times X \end{array} \right\}. \end{aligned}$$

We have

Lemma 3.1 (*Forward recursive formulae*)

$$\begin{aligned} \tilde{\Lambda}_0 &= \tilde{\lambda}_0 \\ \tilde{\Lambda}_{n+1} &= \tilde{\Lambda}_n \vee r_n(X_n) \quad 0 \leq n \leq N-1, \\ \tilde{\Xi}_0 &= \tilde{\xi}_0 \\ \tilde{\Xi}_{n+1} &= \tilde{\Xi}_n \wedge r_n(X_n) \quad 0 \leq n \leq N-1, \\ \Gamma_0 &= \{(\tilde{\lambda}_0, \tilde{\xi}_0)\} \\ \Gamma_{n+1} &= \{(\lambda \vee r_n(x), \xi \wedge r_n(x)) \mid (\lambda, \xi) \in \Gamma_n, x \in X\} \quad 0 \leq n \leq N-1. \end{aligned}$$

Let us now expand the original state space X to a direct product space :

$$Y_n := X \times \Gamma_n \quad 0 \leq n \leq N.$$

We define a sequence of *stop-reward functions* $\{G_n\}_0^N$ by

$$G_n(x; \lambda, \xi) := \lambda \vee g_n(x) - \xi \wedge g_n(x) \quad (x; \lambda, \xi) \in Y_n$$

and a *nonstationary Markov transition law* $q = \{q_n\}_0^{N-1}$ by

$$q_n(y; \mu, \nu | x; \lambda, \xi) := \begin{cases} p(y|x) & \text{if } \lambda \vee r_n(x) = \mu, \quad \xi \wedge r_n(x) = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Let us define $\tilde{\Gamma}_n$ through

$$\tilde{\Gamma}_n := (\tilde{\Lambda}_n, \tilde{\Xi}_n).$$

Then $\{(X_n, \tilde{\Gamma}_n)\}_0^N$ is a Markov chain on state spaces $\{Y_n\}$ with transition law q . We consider the *terminal criterion* $\{G_n\}_0^N$ on the expanded process :

$$\bar{T}_0(y_0) \quad \text{Max } \mathbb{E}_{y_0}[G_\tau] \quad \text{s.t. } \tau \in \tilde{\mathcal{T}}_0^N$$

where $y_0 = (x_0; \tilde{\lambda}_0, \tilde{\mu}_0)$, and $\tilde{\mathcal{T}}_n^N$ is the set of all stopping times which take values in $\{n, n+1, \dots, N\}$ on the new Markov chain.

Now we consider a subprocess which starts at state $y_n = (x_n; \lambda_n, \xi_n) \in Y_n$ on n -th stage :

$$\bar{T}_n(y_n) \quad \text{Max } \mathbb{E}_{y_n}[G_\tau] \quad \text{s.t. } \tau \in \tilde{\mathcal{T}}_n^N.$$

Let $v_n(y_n)$ be the maximum value of $\bar{T}_n(y_n)$, where

$$v_N(y_N) \triangleq G_N(y_N) \quad y_N \in Y_N.$$

Then we have the backward recursive equation :

Corollary 3.1

$$\begin{cases} v_N(y) = G_N(y) & y \in Y_N \\ v_n(y) = \text{Max} [G_n(y), \mathbb{E}_y[v_{n+1}(Y_{n+1})]] & y \in Y_n, \quad 0 \leq n \leq N-1 \end{cases}$$

where \mathbb{E}_y is the one-step expectation operator induced from the Markov transition probabilities $q_n(\cdot|\cdot)$:

$$\mathbb{E}_y[h(Y_{n+1})] = \sum_{z \in Y_{n+1}} h(z) q_n(z|y).$$

Corollary 3.2 The stopping time τ^* :

$$\tau^*(\omega) = \min\{n \geq 0 : v_n(y_n) = G_n(y_n)\} \quad \omega = y_0 y_1 \cdots y_N$$

is optimal :

$$\mathbb{E}_{y_0}[G_{\tau^*}] \geq \mathbb{E}_{y_0}[G_\tau] \quad \forall \tau \in \tilde{\mathcal{T}}_0^N.$$

Then we have the corresponding recursive equation for the original process with maximum reward :

Theorem 3.1

$$\begin{cases} v_N(x; \lambda, \xi) = \lambda \vee g_N(x) - \xi \wedge g_N(x) & x \in X, \quad (\lambda, \xi) \in \Gamma_N \\ v_n(x; \lambda, \xi) = \text{Max} [\lambda \vee g_n(x) - \xi \wedge g_n(x), \mathbb{E}_x[v_{n+1}(X_{n+1}; \lambda \vee r_n(x), \xi \wedge r_n(x))]] & x \in X, \quad (\lambda, \xi) \in \Gamma_n, \quad 0 \leq n \leq N-1. \end{cases} \quad (3)$$

Here we consider a family of subprocesses which start at $x_n (\in X)$ with a pair of accumulated maximum and maximum up to there (λ_n, ξ_n) :

$$\begin{aligned} \text{Ra}_n(x_n; \lambda_n, \xi_n) \quad & \text{Max } E_{x_n}[\lambda_n \wedge r_n \vee \cdots \vee r_{\tau-1} \vee g_\tau - \xi_n \wedge r_n \wedge \cdots \wedge r_{\tau-1} \wedge g_\tau] \\ \text{s.t. } \quad & \tau \in \mathcal{T}_n^N \\ & x_n \in X, (\lambda_n, \xi_n) \in \Gamma_n, 0 \leq n \leq N-1 \end{aligned}$$

where

$$\begin{aligned} & E_{x_n}[\lambda_n \vee r_n \vee \cdots \vee r_{\tau-1} \vee g_\tau - \xi_n \wedge r_n \wedge \cdots \wedge r_{\tau-1} \wedge g_\tau] \\ = & \sum_{m=n}^N \sum_{\{\tau=m\}} \{[\lambda_n \vee r_n(x_n) \vee \cdots \vee r_{m-1}(x_{m-1}) \vee g_m(x_m) \\ & - \xi_n \wedge r_n(x_n) \wedge \cdots \wedge r_{m-1}(x_{m-1}) \wedge g_m(x_m)] \times p(x_{n+1}|x_n)p(x_{n+2}|x_{n+1}) \cdots p(x_m|x_{m-1})\}. \end{aligned}$$

Let $v_n(x_n; \lambda_n, \xi_n)$ be the maximum value for $\text{Ra}_n(x_n; \lambda_n, \xi_n)$, where

$$v_N(x_N; \lambda_N, \xi_N) = \lambda_N \vee g_N(x_N) - \xi_N \wedge g_N(x_N).$$

Then the maximum value functions satisfy the recursive equation (3).

Theorem 3.2 *The stopping time τ^* :*

$$\begin{aligned} \tau^*(\omega) &= \min\{n \geq 0 : v_n(x_n; \lambda_n, \xi_n) = \lambda_n \vee g_n(x_n) - \xi_n \wedge g_n(x_n)\} \\ \omega &= (x_0; \tilde{\lambda}_0, \tilde{\xi}_0)(x_1; \lambda_1, \xi_1) \cdots (x_N; \lambda_N, \xi_N) \end{aligned}$$

is optimal :

$$E_{x_0}[R_{\tau^*}] \geq E_{x_0}[R_{\tau}] \quad \forall \tau \in \mathcal{T}_0^N.$$

3.1 DP solution

Let us illustrate a two-state four-stage model, which is specified by an optimal stopping problem:

$$\begin{aligned} \text{Max } & E_{x_0}[r_0(X_0) \vee \cdots \vee r_{\tau-1}(X_{\tau-1}) \vee g_\tau(X_\tau) - r_0(X_0) \wedge \cdots \wedge r_{\tau-1}(X_{\tau-1}) \wedge g_\tau(X_\tau)] \\ \text{s.t. (i) } & \tau \in \mathcal{T}_0^4 \end{aligned} \quad (4)$$

where the stop/continue-reward $\{g_0, g_1, g_2, g_3, g_4; r_0, r_1, r_2, r_3\}$ is given in Table 1, and the transition matrix is symmetric ($p = q = 1/2$). Let us find an optimal stopping time by solving recursive equation.

Table 1 stop/continue-reward

x_n		s_1	s_2
$g_0(x_0)$	$r_0(x_0)$	5 4	4 5
$g_1(x_1)$	$r_1(x_1)$	6 5	4 6
$g_2(x_1)$	$r_2(x_1)$	3 4	7 6
$g_3(x_1)$	$r_3(x_1)$	5 7	8 3
$g_4(x_4)$		6	7

It is shown that the total number of stopping times $\{f_m(n)\}$ for m -state n -stage model satisfies the recursive relation ([6])

$$\begin{aligned} f_m(n+1) &= 1 + (f_m(n))^m \\ f_2(m) &= 1 + 2^m. \end{aligned}$$

There exist $f_2(4) = 677$ stopping times for two-state ($m = 2$) four-stage ($n = 4$) model. Among them, let us find an optimal stopping time by solving dynamic programming recursive equation (3).

First, the forward recursion in Lemma 3.1 generates the following past-value sets :

$$\begin{aligned} \Gamma_0 &= \{(-\infty, \infty)\}, \quad \Gamma_1 = \{(4, 4)\}, \quad \Gamma_2 = \{(5, 4), (6, 4)\} \\ \Gamma_3 &= \{(5, 4), (6, 4)\}, \quad \Gamma_4 = \{(7, 4), (5, 3), (6, 3)\}. \end{aligned}$$

Second, the backward recursion (3) yields an optimal solution in expanded Markov class $\tilde{\Pi}$; optimal value functions

$$v_0, v_1, v_2, v_3, v_4; \quad v_n = v_n(x_n; \lambda_n, \xi_n)$$

and an optimal policy

$$\pi^* = \{\pi_0^*, \pi_1^*, \pi_2^*, \pi_3^*\}; \quad \pi_n^* = \pi_n^*(x_n; \lambda_n, \xi_n).$$

In fact the optimal solution is calculated as follows:

$$\begin{aligned} v_4(s_1; 7, 4) &= 7 \vee g_4(s_1) - 4 \wedge g_4(s_1) = 7 \vee 6 - 4 \wedge 6 = 7 - 4 = 3 \\ v_4(s_1; 5, 3) &= 5 \vee g_4(s_1) - 3 \wedge g_4(s_1) = 5 \vee 6 - 3 \wedge 6 = 6 - 3 = 3 \\ v_4(s_1; 6, 3) &= 6 \vee g_4(s_1) - 3 \wedge g_4(s_1) = 6 \vee 6 - 3 \wedge 6 = 6 - 3 = 3 \\ v_4(s_2; 7, 4) &= 7 \vee g_4(s_2) - 4 \wedge g_4(s_2) = 7 \vee 7 - 4 \wedge 7 = 7 - 4 = 3 \\ v_4(s_2; 5, 3) &= 5 \vee g_4(s_2) - 3 \wedge g_4(s_2) = 5 \vee 7 - 3 \wedge 7 = 7 - 3 = 4 \\ v_4(s_2; 6, 3) &= 6 \vee g_4(s_2) - 3 \wedge g_4(s_2) = 6 \vee 7 - 3 \wedge 7 = 7 - 3 = 4 \end{aligned}$$

$$\begin{aligned} v_3(s_1; 5, 4) &= \text{Max} \left\{ \begin{array}{l} 5 \vee g_3(s_1) - 4 \wedge g_3(s_1) \\ v_4(s_1; 5 \vee r_3(s_1), 4 \wedge r_3(s_1)) \cdot \frac{1}{2} + v_4(s_1; 5 \vee r_3(s_1), 4 \wedge r_3(s_1)) \cdot \frac{1}{2} \end{array} \right. \\ &= \text{Max} \left\{ \begin{array}{l} 5 \vee g_3(s_1) - 4 \wedge g_3(s_1) \\ v_4(s_1; 5 \vee 7, 4 \wedge 7) \cdot \frac{1}{2} + v_4(s_1; 5 \vee 7, 4 \wedge 7) \cdot \frac{1}{2} \end{array} \right. \\ &= \text{Max} \left\{ \begin{array}{l} 5 \vee 5 - 4 \wedge 5 \\ v_4(s_1; 7, 4) \cdot \frac{1}{2} + v_4(s_1; 7, 4) \cdot \frac{1}{2} \end{array} \right. \\ &= \text{Max} \left\{ \begin{array}{l} 5 - 4 \\ 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} \end{array} \right. = \text{Max} \left\{ \begin{array}{l} 1 \\ 3 \end{array} \right. = 3 \quad \pi_3^*(s_1; 5, 4) = \text{continue} \end{aligned}$$

$$\begin{aligned}
v_3(s_1; 6, 4) &= \text{Max} \left\{ \begin{array}{l} 6 \vee g_3(s_1) - 4 \wedge g_3(s_1) \\ v_4(s_1; 6 \vee r_3(s_1), 4 \wedge r_3(s_1)) \cdot \frac{1}{2} + v_4(s_1; 6 \vee r_3(s_1), 4 \wedge r_3(s_1)) \cdot \frac{1}{2} \end{array} \right. \\
&= \text{Max} \left\{ \begin{array}{l} 6 \vee 5 - 4 \wedge 5 \\ v_4(s_1; 7, 4) \cdot \frac{1}{2} + v_4(s_1; 7, 4) \cdot \frac{1}{2} \end{array} \right. \\
&= \text{Max} \left\{ \begin{array}{l} 6 - 4 \\ 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} \end{array} \right. = \text{Max} \left\{ \begin{array}{l} 2 \\ 3 \end{array} \right. = 3 \quad \pi_3^*(s_1; 6, 4) = \text{continue}
\end{aligned}$$

$$v_3(s_2; 5, 4) = \text{Max} \left\{ \begin{array}{l} 8 - 4 \\ 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} \end{array} \right. = 4 \quad \pi_3^*(s_2; 5, 4) = s$$

$$v_3(s_2; 6, 4) = \text{Max} \left\{ \begin{array}{l} 8 - 4 \\ 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} \end{array} \right. = 4 \quad \pi_3^*(s_2; 6, 4) = s$$

$$v_2(s_1; 5, 4) = \text{Max} \left\{ \begin{array}{l} 5 - 3 \\ 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} \end{array} \right. = 3.5 \quad \pi_2^*(s_1; 5, 4) = c$$

$$v_2(s_1; 6, 4) = \text{Max} \left\{ \begin{array}{l} 6 - 3 \\ 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} \end{array} \right. = 3.5 \quad \pi_2^*(s_1; 6, 4) = c$$

$$v_2(s_2; 5, 4) = \text{Max} \left\{ \begin{array}{l} 7 - 4 \\ 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} \end{array} \right. = 3.5 \quad \pi_2^*(s_2; 5, 4) = c$$

$$v_2(s_2; 6, 4) = \text{Max} \left\{ \begin{array}{l} 7 - 4 \\ 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} \end{array} \right. = 3.5 \quad \pi_2^*(s_2; 6, 4) = c$$

$$v_1(s_1; 4, 4) = \text{Max} \left\{ \begin{array}{l} 6 - 4 \\ 3.5 \cdot \frac{1}{2} + 3.5 \cdot \frac{1}{2} \end{array} \right. = 3.5 \quad \pi_1^*(s_1; 4, 4) = c$$

$$v_1(s_2; 4, 4) = \text{Max} \left\{ \begin{array}{l} 4 - 4 \\ 3.5 \cdot \frac{1}{2} + 3.5 \cdot \frac{1}{2} \end{array} \right. = 3.5 \quad \pi_1^*(s_2; 4, 4) = c$$

$$v_0(s_1; -\infty, \infty) = \text{Max} \left\{ \begin{array}{l} 5 - 5 \\ 3.5 \cdot \frac{1}{2} + 3.5 \cdot \frac{1}{2} \end{array} \right. = 3.5 \quad \pi_0^*(s_1; -\infty, \infty) = c$$

The optimal solution is tabulated in Table 2:

Table 2 optimal solution

$x_4 \setminus (\lambda_4, \xi_4)$	$v_4(x_4; \lambda_4, \xi_4)$		
	(7,4)	(5,3)	(6,3)
s_1	3	3	3
s_2	3	4	4

$x_n \setminus (\lambda_n, \xi_n)$	$v_3(x_3; \lambda_3, \xi_3) \quad \pi_3^*(x_3; \lambda_3, \xi_3)$		$v_2(x_2; \lambda_2, \xi_2) \quad \pi_2^*(x_2; \lambda_2, \xi_2)$	
	(5,4)	(6,4)	(5,4)	(6,4)
s_1	3 c	3 c	3.5 c	3.5 c
s_2	4 s	4 s	3.5 c	3.5 c

$x_n \setminus (\lambda_n, \xi_n)$	$v_1(x_1; \lambda_1, \xi_1) \pi_1^*(x_1; \lambda_1, \xi_1)$	$v_0(x_0; 1) \pi_0^*(x_0; \lambda_0, \xi_0)$
	(4,4)	$(-\infty, \infty)$
s_1	3.5 c	3.5 c
s_2	3.5 c	— —

Third, we see that an optimal stopping rule τ^* is to stop at state s_2 on stage 3. The rule implies that an optimal decision-maker should continue at any state on stages 0, 1, 2 and at state s_1 on stage 3 (Figure 1). The maximum expected value of range is $v_0(s_0; -\infty, \infty) = 3.5$. As is directly verified at the bottom line, the optimal expected value is equal to $E_{s_1}[R_{\tau^*}]$. The stopping time τ^* has

$$E_{s_1}[R_{\tau^*}] = 3 \cdot \frac{1}{16} + 3 \cdot \frac{1}{16} + \cdots + 4 \cdot \frac{1}{8} = \frac{7}{2} = 3.5.$$

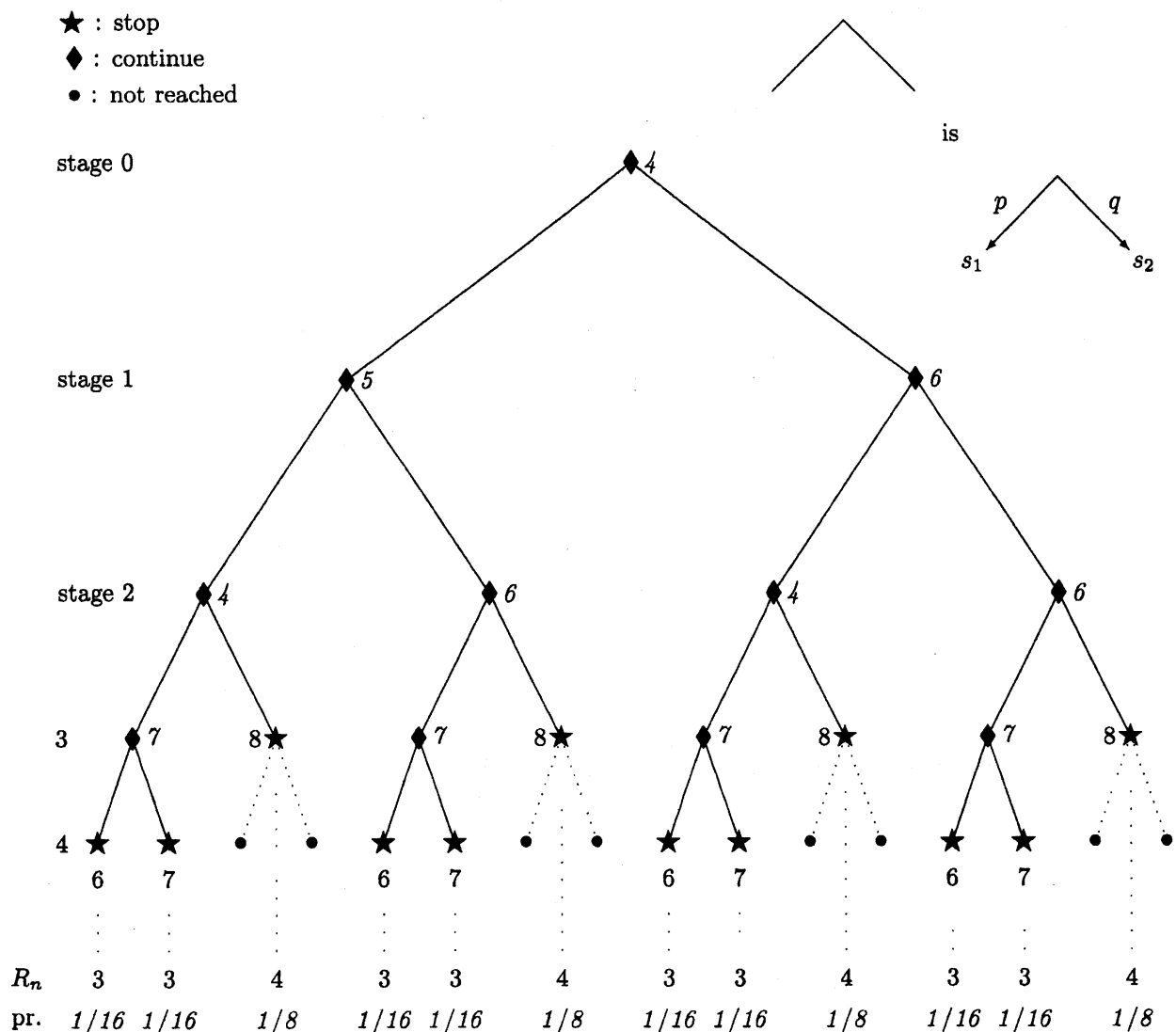


Figure 1 It is optimal to stop at state s_2 on stage 3 : τ^*

References

- [1] R.E. Bellman, *Dynamic Programming*, Princeton Univ. Press, NJ, 1957.

- [2] R.E. Bellman and E.D. Denman, *Invariant Imbedding*, Lect. Notes in Operation Research and Mathematical Systems, Vol. 52, Springer-Verlag, Berlin, 1971.
- [3] R.E. Bellman and L.A. Zadeh, Decision-making in a fuzzy environment, *Management Sci.* **17**(1970), B141-B164.
- [4] Y.S. Chow, H. Robbins and D. Siegmund, *Great Expectations: The Theory of Optimal Stopping*, Houghton Mifflin Company, Boston, 1971.
- [5] N. Furukawa and S. Iwamoto, Stopped decision processes on complete separable metric spaces, *J. Math. Anal. Appl.* **31**(1970), 615-658.
- [6] H. Hisano, Optimal stopping problem on finite Markov chain, *Bull. Informatics and Cybernetics*, **34**, No. 2 (2003), 97-104.
- [7] S. Iwamoto, *Theory of Dynamic Program: Japanese*, Kyushu Univ. Press, Fukuoka, 1987.
- [8] S. Iwamoto, From dynamic programming to bynamic programming, *J. Math. Anal. Appl.* **177**(1993), 56-74.
- [9] S. Iwamoto, On bidecision processes, *J. Math. Anal. Appl.* **187**(1994), 676-699.
- [10] S. Iwamoto, Associative dynamic programs, *J. Math. Anal. Appl.*, **201**(1996), 195-211.
- [11] S. Iwamoto, Maximizing threshold probability through invariant imbedding, Ed. H.F. Wang and U.P. Wen, Proceedings of the 8th Bellman Continuum, National Tsing Hua University, Hsinchu, ROC, Dec., 2000, 17-22.
- [12] S. Iwamoto, Fuzzy decision-making through three dynamic programming approaches, d. H.F. Wang and U.P. Wen, Proceedings of the 8th Bellman Continuum, National Tsing Hua University, Hsinchu, ROC, Dec., 2000, 23-27.
- [13] S. Iwamoto, Recursive method in stochastic optimization under compound criteria, *Advances in Mathematical Economics* **3**(2001), 63-82.
- [14] S. Iwamoto, K. Tsurusaki and T. Fujita, "On Markov policies for minimax decision processes," *J. Math. Anal. Appl.*, vol.253, no.1, pp.58-78, 2001.
- [15] S. Iwamoto, T. Ueno and T. Fujita, "Controlled Markov chains with utility functions," *Proc. of Intl Workshop on Markov Processes and Controlled Markov Chains*, Changsha, China, August 22-28, 1999, Kluwer, 2002, pp.135-148. to appear.
- [16] S. Iwamoto and T. Fujita, Stochastic decision-making in a fuzzy environment, *J. Operations Res. Soc. Japan* **38**(1995), 467-482.
- [17] E.S. Lee, *Quasilinearization and Invariant Imbedding*, Academic Press, New York, 1968.
- [18] A.N. Shiryaev, *Optimal Stopping Rules*, Springer-Verlag, New York, 1978.
- [19] M. Sniedovich, *Dynamic Programming*, Marcel Dekker, Inc. NY, 1992.
- [20] K. Tsurusaki, Extrema-trimmed sum in decision problem, Ed. H.F. Wang and U.P. Wen, Proceedings of the 8th Bellman Continuum, National Tsing Hua University, Hsinchu, ROC, Dec., 2000, 2-6.